Specialized Numerical Methods for Transport Phenomena

The finite element method: Poisson problem

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Outline

Recapitulation

Finite Element Method: Developing a 1D understanding

A complete example by hand

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Last few courses

In the last few courses, we have introduced (or reviewed for many) the following concepts:

- Triangulating a domain (generating a mesh)
- Integrating over a domain
- Interpolating over a domain

We will now use all of these notions together to solve problems using the Finite Element method.

Type of problems

In the next 6 courses, we will look at problems of increasing complexity

- Linear (Poisson) problem such as the heat equation (HW3)
- Advection-Diffusion problems and the notion of upwinding (HW4)
- Non-linear problems (HW4)
- Navier-Stokes equations (HW5)

After this, we will have acquired a good understanding of a classical numerical method and we will be able to move on to the specialized methods.

Outline

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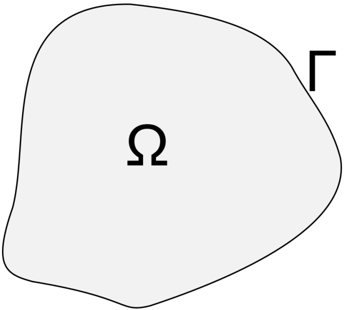
A complete example by hand

The heat equation, a PDE prototype

We are interested in solving equations such as the heat equation on a Ω domain whose contour is Γ:

$$\nabla^2 T = 0$$



1D version of the problem

Let's start by solving the 1D heat equation, with Dirichlet boundary conditions:

$$\frac{\mathrm{d}^2 T}{\mathrm{d}x^2} = 0$$

We can multiply our equation by an a priori unknown function $u(x)$, as long as it is bounded and obtain:

$$\frac{\mathrm{d}^2 T}{\mathrm{d}x^2} u(x) = 0$$

We can integrate this equation over the entire domain of interest and obtain the **strong integral form**:

$$\int_\Omega \frac{\mathrm{d}^2 T}{\mathrm{d}x^2} u(x) \mathrm{d}x = 0$$

The role of interpolation

$$\int_\Omega \frac{\mathrm{d}^2T}{\mathrm{d}x^2}u(x)\mathrm{d}x = 0$$

Let $\Omega = [0, L]$ be an interval partitioned by $n+1$ points $\{x_i\}_{i=0}^n$ defining the mesh Ω_h. We represent the temperature $T(x)$ using a polynomial space made of our continuous piecewise Lagrange polynomial. For example, we choose piecewise linear Lagrange polynomials.
Now our temperature is:

$$T(x) = \sum_{j=0}^{n} T_j\varphi_j(x)$$

The role of interpolation

Since we know that $T(x)$ is now:

$$T(x) = \sum_{j=0}^{n} T_j \varphi_j(x)$$

We can replace it in the strong integral form:

$$\int_{\Omega} \frac{\mathrm{d}^2 T}{\mathrm{d}x^2} u(x) \mathrm{d}x = 0$$

The role of interpolation

Since we know that $T(x)$ is now:

$$T(x) = \sum_{j=0}^{n} T_j \varphi_j(x)$$

We can replace it in the strong integral form:

$$\int_\Omega \frac{\mathrm{d}^2 T}{\mathrm{d}x^2} u(x) \mathrm{d}x = 0$$

Which now reads:

$$\int_\Omega \frac{\mathrm{d}^2 \sum_{j=0}^{n} (T_j \varphi_j)}{\mathrm{d}x^2} u(x) \mathrm{d}x = 0$$

$$\int_\Omega \sum_{j=0}^{n} T_j \frac{\mathrm{d}^2 \varphi_j}{\mathrm{d}x^2} u(x) \mathrm{d}x = 0$$

Towards weak form

$$\int_{\Omega} \sum_{j=0}^{n} T_j \frac{\mathrm{d}^2 \varphi_j}{\mathrm{d}x^2} u(x) \mathrm{d}x = 0$$

We have a problem! $\varphi_j(x)$ being linear, $\frac{\mathrm{d}^2 \varphi_j(x)}{\mathrm{d}x^2} = 0$.
What to do?

Towards weak form

$$\int_\Omega \sum_{j=0}^{n} T_j \frac{\mathrm{d}^2\varphi_j}{\mathrm{d}x^2} u(x)\mathrm{d}x = 0$$

By performing an integral by parts, we can go from the **strong integral form** to the **weak integral form**!

$$\int_\Omega \sum_{j=0}^{n} T_j \frac{\mathrm{d}^2\varphi_j}{\mathrm{d}x^2} u(x)\mathrm{d}x = \left[\sum_{j=0}^{n} T_j \frac{\mathrm{d}\varphi_j}{\mathrm{d}x} u(x)\right]_{x=L} - \left[\sum_{j=0}^{n} T_j \frac{\mathrm{d}\varphi_j}{\mathrm{d}x} u(x)\right]_{x=0} - \int_\Omega \sum_{j=0}^{n} T_j \frac{\mathrm{d}\varphi_j}{\mathrm{d}x} \frac{\mathrm{d}u(x)}{\mathrm{d}x}\mathrm{d}x = 0$$

Remember that this equation is valid for all $u(x) \in \mathcal{L}^2$.

Boundary terms

$$\left[\sum_{j=0}^{n} T_j \frac{\mathrm{d}\varphi_j}{\mathrm{d}x} u(x)\right]_{x=L} - \left[\sum_{j=0}^{n} T_j \frac{\mathrm{d}\varphi_j}{\mathrm{d}x} u(x)\right]_{x=0} - \int_{\Omega} \sum_{j=0}^{n} T_j \frac{\mathrm{d}\varphi_j}{\mathrm{d}x} \frac{\mathrm{d}u(x)}{\mathrm{d}x} \mathrm{d}x = 0$$

When we have Dirichlet conditions, we know the value of the temperature on Γ. Thus we can choose $u(x)$ such that $u(x)$ is zero on the boundary:

$$u(0) = 0$$
$$u(L) = 0$$

We then obtain for the interior of the domain:

$$\int_{\Omega} \sum_{j=0}^{n} T_j \frac{\mathrm{d}\varphi_j}{\mathrm{d}x} \frac{\mathrm{d}u(x)}{\mathrm{d}x} \mathrm{d}x = 0$$

Inside the domain

$$\int_\Omega \sum_{j=0}^{n} T_j \frac{\mathrm{d}\varphi_j}{\mathrm{d}x} \frac{\mathrm{d}u(x)}{\mathrm{d}x} \mathrm{d}x = 0$$

We need to choose the test function $u(x)$ inside the domain. In Galerkin type finite element, we choose the test function identical to the interpolation function. So we get $n-1$ equations of the form:

$$\int_\Omega \sum_{j=0}^{n} T_j \frac{\mathrm{d}\varphi_j}{\mathrm{d}x} \frac{\mathrm{d}\varphi_i}{\mathrm{d}x} = 0$$

Final problem

$$\int_\Omega \sum_{j=0}^{n} T_j \frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\frac{\mathrm{d}\varphi_i}{\mathrm{d}x}\mathrm{d}x = 0 \quad \forall i \in [1, n-1]$$

$$T_0 = T(x=0)$$
$$T_n = T(x=L)$$

Steps of the resolution:

- Define the triangulation and the elements (Ω_h)
- Define the interpolation functions (φ_i) and their gradient $\left(\frac{\mathrm{d}\varphi_i}{\mathrm{d}x}\right)$
- Define the structure of the matrix
- Calculate the integral to calculate the matrix (e.g., $\int_{\Omega_1} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}$)
- Solve the linear system of equations to find the T_j
- The temperature is now known everywhere because of the interpolation support!

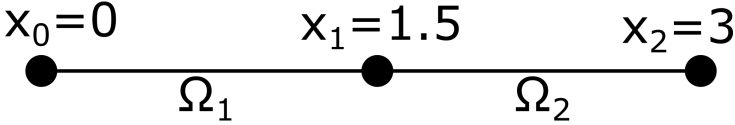
Outline

Example using two elements

Let us assume a rod of length $L = 3\,$m with the temperature on the left fixed at $T = 10\,^\circ$C and the temperature on the right fixed at $T = 20\,^\circ$C. to $T = 20\,^\circ$C. Calculate the temperature profile with the finite element method using two elements of equal size.

Defining the intervals

We use two elements (Ω_1 and Ω_2), so three nodes (x_0,x_1,x_2). The corresponding mesh is:



The size of each element is the same, we will note it h.

Definition of the interpolation functions

We have three interpolation functions φ_0, φ_1 and φ_2:

$$\Omega_1 \begin{cases} \varphi_0 = \frac{x_1 - x}{x_1 - x_0} \\ \varphi_1 = \frac{x - x_0}{x_1 - x_0} \\ \varphi_2 = 0 \end{cases} \quad \Omega_2 \begin{cases} \varphi_0 = 0 \\ \varphi_1 = \frac{x_2 - x}{x_2 - x_1} \\ \varphi_2 = \frac{x - x_1}{x_2 - x_1} \end{cases}$$

As we have elements that are all the same size we can rewrite in the form:

$$\Omega_1 \begin{cases} \varphi_0 = \frac{x_1 - x}{h} \\ \varphi_1 = \frac{x - x_0}{h} \\ \varphi_2 = 0 \end{cases} \quad \Omega_2 \begin{cases} \varphi_0 = 0 \\ \varphi_1 = \frac{x_2 - x}{h} \\ \varphi_2 = \frac{x - x_1}{h} \end{cases}$$

Gradients of interpolation functions

$$\Omega_1 \begin{cases} \varphi_0 = \frac{x_1 - x}{h} \\ \varphi_1 = \frac{x - x_0}{h} \\ \varphi_2 = 0 \end{cases} \quad \Omega_2 \begin{cases} \varphi_0 = 0 \\ \varphi_1 = \frac{x_2 - x}{h} \\ \varphi_2 = \frac{x - x_1}{h} \end{cases}$$

We need to calculate the gradient of the interpolation function $\left(\frac{\mathrm{d}\varphi_i}{\mathrm{d}x}\right)$:

$$\Omega_1 \begin{cases} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x} = -\frac{1}{h} \\ \frac{\mathrm{d}\varphi_1}{\mathrm{d}x} = \frac{1}{h} \\ \frac{\mathrm{d}\varphi_2}{\mathrm{d}x} = 0 \end{cases} \quad \Omega_2 \begin{cases} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x} = 0 \\ \frac{\mathrm{d}\varphi_1}{\mathrm{d}x} = -\frac{1}{h} \\ \frac{\mathrm{d}\varphi_2}{\mathrm{d}x} = \frac{1}{h} \end{cases}$$

Building the linear system of equations

We discretize Ω with three points, so we have φ_0, φ_1 et φ_2.

$$\int_\Omega \sum_{j=0}^{2} T_j \frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\frac{\mathrm{d}\varphi_i}{\mathrm{d}x}\mathrm{d}x = 0$$

Becomes three equations:

$$\int_\Omega \sum_{j=0}^{2} T_j \frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\mathrm{d}x = 0$$

$$\int_\Omega \sum_{j=0}^{2} T_j \frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\mathrm{d}x = 0$$

$$\int_\Omega \sum_{j=0}^{2} T_j \frac{\mathrm{d}\varphi_j}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\mathrm{d}x = 0$$

Expanding the last summation

The last sum is expanded:

$$\int_{\Omega} T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} + T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\mathrm{d}x = 0$$

$$\int_{\Omega} T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\mathrm{d}x = 0$$

$$\int_{\Omega} T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x} + T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\mathrm{d}x = 0$$

What's left to do?

- Take care of the integral
- Solve a linear system of equations to obtain the T_j and obtain the temperature profile!

Expanding the integral

We solve with 3 points, we have two intervals $\Omega_1 = [x_0, x_1]$ and $\Omega_2 = [x_1, x_2]$. We obtain:

$$\begin{aligned}
&\int_{\Omega_1} T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} + T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\mathrm{d}x \\
+&\int_{\Omega_2} T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} + T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\mathrm{d}x = 0 \\
&\int_{\Omega_1} T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\mathrm{d}x \\
+&\int_{\Omega_2} T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\mathrm{d}x = 0 \\
&\int_{\Omega_1} T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x} + T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\mathrm{d}x \\
+&\int_{\Omega_2} T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x} + T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\mathrm{d}x = 0
\end{aligned}$$

Simplifying

$$
\begin{aligned}
&\int_{\Omega_1} T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} \cancel{+ T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x}} \mathrm{d}x \\
+ &\int_{\Omega_2} \cancel{T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} + T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x}} \mathrm{d}x = 0 \\
&\int_{\Omega_1} T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + \cancel{T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}} \mathrm{d}x \\
+ &\int_{\Omega_2} \cancel{T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} \mathrm{d}x = 0 \\
&\int_{\Omega_1} \cancel{T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x} + T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x}} \mathrm{d}x \\
+ &\int_{\Omega_2} \cancel{T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x}} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x} + T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x} \mathrm{d}x = 0
\end{aligned}
$$

Result

$$\int_{\Omega_1} T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\mathrm{d}x = 0$$

$$\int_{\Omega_1} T_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\mathrm{d}x + \int_{\Omega_2} T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\mathrm{d}x = 0$$

$$+ \int_{\Omega_2} T_1 \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x} + T_2 \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\mathrm{d}x = 0$$

In matrix-vector form:

$$\begin{bmatrix} \int_{\Omega_1} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} & \int_{\Omega_1} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} & 0 \\ \int_{\Omega_1} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} & \int_{\Omega_1} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + \int_{\Omega_2} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} & \int_{\Omega_2} \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} \\ 0 & \int_{\Omega_2} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x} & \int_{\Omega_2} \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x} \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Dirichlet boundary conditions

$$\begin{bmatrix} \int_{\Omega_1} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} & \int_{\Omega_1} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} & 0 \\ \int_{\Omega_1} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} & \int_{\Omega_1} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + \int_{\Omega_2} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} & \int_{\Omega_2} \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} \\ 0 & \int_{\Omega_2} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x} & \int_{\Omega_2} \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x} \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We impose the Dirichlet boundary conditions **strongly** by imposing that the temperature at the boundary respects the boundary condition

$$\begin{bmatrix} 1 & 0 & 0 \\ \int_{\Omega_1} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} & \int_{\Omega_1} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + \int_{\Omega_2} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} & \int_{\Omega_2} \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 10\,^{\circ}\mathrm{C} \\ 0 \\ 20\,^{\circ}\mathrm{C} \end{bmatrix}$$

Calculating the integrals

$$
\begin{bmatrix}
1 & 0 & 0 \\
\int_{\Omega_1} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} & \int_{\Omega_1} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + \int_{\Omega_2} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} & \int_{\Omega_2} \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} T_0 \\ T_1 \\ T_2 \end{bmatrix}
=
\begin{bmatrix} 10\,^\circ\mathrm{C} \\ 0 \\ 20\,^\circ\mathrm{C} \end{bmatrix}
$$

$$
\Omega_1 \begin{cases} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x} = -\frac{1}{h} \\ \frac{\mathrm{d}\varphi_1}{\mathrm{d}x} = \frac{1}{h} \\ \frac{\mathrm{d}\varphi_2}{\mathrm{d}x} = 0 \end{cases}
\qquad
\Omega_2 \begin{cases} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x} = 0 \\ \frac{\mathrm{d}\varphi_1}{\mathrm{d}x} = -\frac{1}{h} \\ \frac{\mathrm{d}\varphi_2}{\mathrm{d}x} = \frac{1}{h} \end{cases}
$$

Now we just have to calculate the value of the gradients and integrate!

Overview

$$
\begin{bmatrix}
1 & 0 & 0 \\
\int_{\Omega_1} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} & \int_{\Omega_1} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + \int_{\Omega_2} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} & \int_{\Omega_2} \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} T_0 \\ T_1 \\ T_2 \end{bmatrix}
=
\begin{bmatrix} 10\,^\circ\mathrm{C} \\ 0 \\ 20\,^\circ\mathrm{C} \end{bmatrix}
$$

$$
\Omega_1 \begin{cases} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x} = -\frac{1}{h} \\ \frac{\mathrm{d}\varphi_1}{\mathrm{d}x} = \frac{1}{h} \\ \frac{\mathrm{d}\varphi_2}{\mathrm{d}x} = 0 \end{cases}
\qquad
\Omega_2 \begin{cases} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x} = 0 \\ \frac{\mathrm{d}\varphi_1}{\mathrm{d}x} = -\frac{1}{h} \\ \frac{\mathrm{d}\varphi_2}{\mathrm{d}x} = \frac{1}{h} \end{cases}
$$

$$
\int_{\Omega_1} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\mathrm{d}x = -\frac{1}{h} \quad , \quad \int_{\Omega_1} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\mathrm{d}x = \frac{1}{h}
$$

$$
\int_{\Omega_2} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\mathrm{d}x = \frac{1}{h} \quad , \quad \int_{\Omega_2} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\mathrm{d}x = -\frac{1}{h}
$$

Result

$$\int_{\Omega_1} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} = -\frac{1}{h} \ , \ \int_{\Omega_1} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} = \frac{1}{h}$$

$$\int_{\Omega_2} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} = \frac{1}{h} \ , \ \int_{\Omega_2} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_2}{\mathrm{d}x} = -\frac{1}{h}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \int_{\Omega_1} \frac{\mathrm{d}\varphi_0}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} & \int_{\Omega_1} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + \int_{\Omega_2} \frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} & \int_{\Omega_2} \frac{\mathrm{d}\varphi_2}{\mathrm{d}x}\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 10\,^\circ\mathrm{C} \\ 0 \\ 20\,^\circ\mathrm{C} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 10\,^\circ\mathrm{C} \\ 0 \\ 20\,^\circ\mathrm{C} \end{bmatrix}$$

We will obtain $T_1 = 15$!

Comments

That was hard and tedious!

Hopefully, from this example you will have seen how hard and tedious FEM is to use manually! There is a reason why we use a computer to solve this problems!

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Hopefully, from this example you will have seen how hard and tedious FEM is to use manually! There is a reason why we use a computer to solve this problems!

Next class

- We will learn to generalize what we have seen above to 2D and 3D problems
- We will discuss the basics of linear algebra to solve the large matrices that arise
- We will learn how to code a solver!