GCH8108E
HW4 - Transport and non-linear problems
February 13, 2025

This homework consists of two problems and will count towards 10% of your total grade for the semester. You need to hand in your code (in a single .zip file and without build files) and a LaTeX report describing the solution of each problem (use the template provided). **Submission deadline:** Monday, February 24th before the class (12h45).

1 Transport problem

We have seen in class that the solution of transport problem using FEM is not a trivial endeavour. Challenges mainly occur for high values of the Péclet number. Experience has taught the author of this homework that this is, regrettably, the case that is often seen in industrial applications. Flows with low Péclet number are generally the exception, not the norm...

In this question, we will solve a 1D and 2D canonical advection-diffusion problem and measure the behavior of the solution as a function of the mesh refinement and the Péclet number. The code template possesses an option to enable or disable stabilization. We have alredy coded for you the SUPG stabilization for the problem your are simulating, thus you only need to code the weak-form for the steady-state advection-diffusion problem. Please take the time to understand the underlying code.

1.1 1D problem

The first problem is the 1D advection diffusion equation:

$$u\frac{\mathrm{d}c}{\mathrm{d}x} - \epsilon\frac{\mathrm{d}^2 c}{\mathrm{d}x^2} = 0 \tag{1}$$

with $\epsilon = \frac{1}{\mathrm{Pe}}$ and $u = 1$. We simulate a domain Ω such that $x \in [0, 1]$ and with Dirichlet boundary conditions such that $c(x = 0) = 0$ and $c(x = 1) = 1$. On this domain, the problem has the following analytical solution:

$$c = \frac{\exp\left(\mathrm{Pe}x - \mathrm{Pe}\right) - \frac{1}{\exp\mathrm{Pe}}}{1 - \frac{1}{\exp\mathrm{Pe}}} \tag{2}$$

There is a reason we have rearranged the analytical solution in this fashion. Writing it in the traditional form:

$$c = \frac{\exp\left(\mathrm{Pe}x\right) - 1}{\exp\mathrm{Pe} - 1} \tag{3}$$

leads to a `inf` over `inf` floating point error, as the author of this homework has learnt the hard way.

Complete the advection-diffusion code template to solve this problem.

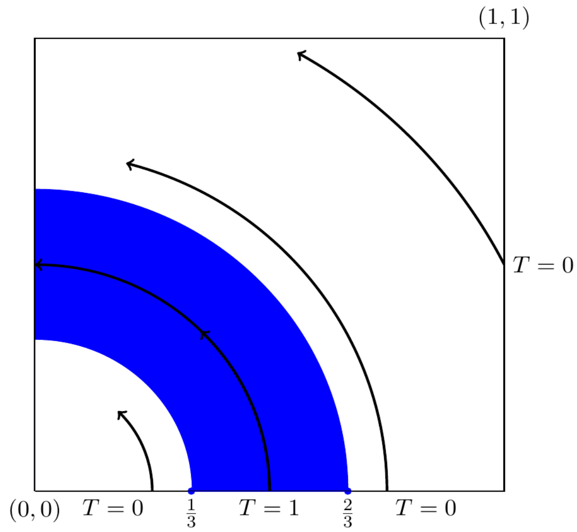

Figure 1: Circular advection-diffusion problem with corresponding boundary conditions.

(a) Write the weak form of the non-stabilized formulation.

(b) Simulate this problem using a refinement level of 5 and Péclet numbers of $[100, 1000, 10000]$. Plot the results. Comment on what you observe.

(c) For a Péclet number of 1000, simulate this problem with refinement levels of 6, 8 and 10. Plot the results and comment on them.

(d) Repeat the exercise of (b), but enable the stabilization. Compare the results.

(e) Compare the convergence as function of the mesh size h for both the stabilized and the non-stabilized version of the code for a Péclet number of 1000. Go to very fine refinement level (e.g. 15). Is the order of the underlying scheme preserved by the stabilization?

1.2 2D problem

Stabilization is not perfect! Oscillations in the direction orthogonal to the advection can occur in 2D and 3D. They are called cross-wind oscillations. We will witness those oscillations using a canonical test case. We consider a square domain such that $x \in [0, 1]$ and $y \in [0, 1]$ illustrated at Figure 1. We solve the advection-diffusion equation for temperature T:

$$\boldsymbol{v} \cdot \nabla T - \epsilon \nabla^2 T = 0 \tag{4}$$

with $\boldsymbol{v} = [-y, x]$ and $\epsilon = 10^{-4}$.

(a) By adapting the code, solve the problem for $\epsilon = 10^{-4}$ and illustrate the results obtained by using a `Warp by Scalar` representation in Paraview. Comment on the results.

(b) Compare the results obtained with and without stabilization. Are there any significant differences? Choose an adequate visualization method.

2 A simple non-linear Poisson problem

In this problem, we investigate the heat dissipation from a solid plate by the combined action of conduction and radiation. The equation we solve is:

$$k\nabla^2 T - \sigma\epsilon\left(T^4 - T_\infty^4\right) = 0 \tag{5}$$

on a domain Ω which is a 2D hyper shell with an inner radius of $R_i = 1$ and outer radius $R_O = 3$. ϵ is the emissivity, which we will take a value equal to 1, k is the thermal conductivity, which we will also be equal to 1, and $\sigma = 5.67\cdot 10^{-8}$ is a constant. The problem is solved with one Dirichlet boundary condition such that $T(r = R_i) = 500\,\mathrm{K}$ and a Neumann boundary condition such that $\nabla T(r = R_O) \cdot \boldsymbol{n} = 0$. We solve this problem in Cartesian coordinates. We will solve this problem using Newton's method.

(a) What is the Jacobian matrix associated with this problem? Demonstrate its origin.

(b) By completing the template code, solve the problem using Newton's method. Illustrate the results you obtain. How many iterations are required to reach a tolerance of $1e^{-10}$?

(c) Plot the evolution of an adequate norm for the correction vector as a function of the number of Newton iteration. Conclude on how many iterations are actually required to reach an accuracy of 0.1 on the temperature field.