GCH8108E
HW2 - Meshes and integration in `deal.II`
January 22, 2025

This homework will count towards 10% of your total grade for the semester. You need to hand in your code (in a single .zip file and without build files) and a LaTeX report describing the solution of each problem (use the template provided). **Submission deadline:** Thursday, January 30th before the class (13h45).

1 Generate, read and refine a mesh

In `deal.II` an object called Triangulation is in charge of storing geometric and topological properties of a mesh. A mesh is essential for any numerical method. Therefore, in this first exercise, you will learn how to generate and refine a mesh in `deal.II`, and how to read an external mesh using the template code `meshes.cc`.

In `deal.II` one can generate simple meshes such as hypercubes (squares, cubes) and hyperballs (circles, spheres). These meshes are useful for problems with simple geometries and they are also commonly used to test different linear algebra solvers. In this problem we will create two meshes: i) a quadrant (in 2D) and ii) an annulus (in 3D).

To obtain a good numerical solution, the size of the elements is often an important parameter. While we would like to always have a very refined mesh, this is not always the best idea as we have limited resources in terms of memory and computing time. Therefore, refining the mesh only in certain areas that are relevant according to the behavior of the solution is very useful. To test how refinement works, after generating the grids both will be refined by two different criteria:

- **Quadrant:** Create a coarse mesh defined by $[-1, 1]^2$. Refine this mesh uniformly 3 times. Then refine 4 times all the cells in the negative quadrant of the geometry.
- **Annulus:** Create a coarse mesh defined by $[-1, 1]^3$. Refine this mesh uniformly 5 times. Then, in order:
  - Refine once all cells whose center lies in the sphere of radius 0.55.
  - Refine once all cells in the shell between radius 0.3 and 0.43.
  - Refine once all cells in the shell between radius 0.335 and 0.39.

One can also set important information for each cell, e.g., `boundary_id` and `material_id`. The latter is useful, for example, when we are simulating different materials and we need to differentiate parts of the domain. To test this, we will change the `material_id` of some cells and visualize it.

Furthermore, most of the problems in practice have a difficult geometry. In this case, the deal.II built-in grid generator is not enough. Usually, a geometry file is generated and then meshed using other software (e.g., Gmsh). In this second part of the exercise, we will import a mesh file generated by gmsh in deal.II. The mesh file corresponds to the `cylinder-structured.msh` file provided. Be careful when specifying the path to the mesh file. This mesh corresponds in fact, to the mesh used to run the simulation of the last exercise of the previous homework.

In your report show screenshots of each of the meshes where the refinement and the `material_id` (if needed) can be properly identified, and show the main information of each of the meshes obtained.

2 Integration in 2D and 3D

The discretization of a PDE using the finite element method leads eventually to integrals. However, the analytical solution of integrals is only known for a few functions. In general, it is not possible to compute it and we need to approximate it numerically. This is when we use numerical quadrature. In FEM, the quadrature is used to integrate matrix entries and components of the right-hand side vector. However, we are not there yet, first, we will use the deal.II quadrature infrastructure to integrate simple functions.

To do so, familiarize yourself first with the following three classes of the library `QGauss<dim>`, `FEValues<dim>` and `FE_Q<dim>`. These are the essential objects to use quadrature to calculate integrals over the triangulation. Due to its templated code and good implementation, `deal.II` enables us to calculate integrals by quadrature in different dimensions. Follow the instructions in the `integration.cc` file to implement three different integration routines:

- **1D:** Integrate the function $f(x) = \sin(x)$ in the interval $[0, 1]$. Compare the result with the analytical solution. Note that this is the same function integrated in the previous homework.

- **2D:** Integrate the function $f(x) = \sin(x)\sin(y)$ in the domain given by $[0, \pi]^2$.

- **3D:** Integrate the function $f(x) = xyz$ in the interval $[0, 1]^3$.

In your report explain the steps to calculate an integral using the special objects of `deal.II` for any dimension. For the 1D case above, present (graphically) the evolution of the error as a function of the mesh size when 2, 3 or 4 quadrature points are used and analyze.